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CLASSIFICATION OF BOUND AND UNBOUND GEODESICS IN THE KERR METRIC AND EFFECTIVE PARTICLE CROSS-SECTION OF A REISSNER - NORDSTROM BLACK HOLE

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ABSTRACT. *Sets of motion constants of a particle that correspond to different types of r-motion are considered. The topology of these sets is determined and a number of constants characterizing these sets are found. An expression is obtained for the capture cross-section of a particle moving at an arbitrary velocity at infinity in the gravitational field of the Schwarzschild black hole. Slow and ultrarelativistic particles are considered. A comparison is made with the results of Meilnik and Plebansky. We examine possible finite motions of particles in the Kerr metric. We determine the topology of various sets of motion constants of the particle and identify invariants that are independent of the rotation parameter. An expression is obtained for the photon capture cross-section of the Reissner-Nordstrom's black hole. An expression is obtained for the uncharged slow particle cross-section of the Reissner-Nordstrom black hole. The Schwarzschild black hole and extreme Reissner-Nordstrom black hole are considered.*

1. TYPES OF UNBOUND GEODESICS IN THE KERR METRIC

An important problem in the study of unbound motion of particles in the Kerr metric is the description of the set of constants of motion for which a particle traveling from infinity goes below the horizon of a black hole. We shall give a qualitative description of this set and also of the set of constants of motion for which the particle asymptotically approaches a surface r = const, placed around the black hole, and the sets of constants of motion for which the particle departs to infinity. The solution to this problem is important in connection with the accretion of non-interacting particles onto a rotating black hole.

The equation of motion for the radial variable in the Kerr metric is (Chandrasekhar, 1983):

$$\rho^4 (dr/d\tau)^2 = R(r), \tag{1}$$

$$R(r) = r^4+(a^2-\xi^2-\eta)r^2 + 2M[\eta + (\xi-a)^2]r - a^2\eta \text{ (Photons)},$$

$$R(r) = r^4+(a^2-\xi^2-\eta)r^2 + 2M[\eta + (\xi-a)^2]r - a^2\eta - r^2\Delta/E \text{ (Particles)},$$

where

$$\rho^2 = r^2 + a^2\cos^2\theta, \ \Delta = r^2-2Mr + a^2, \ a = S/M. \tag{2}$$

The constants S and M refer to the black hole; S is the angular momentum and M is the mass of the black hole. The constants E, ξ and η refer to the particle, namely E is its energy at infinity, $\xi = L_z/E$ (L_z is the angular momentum of the particle about the rotation axis of the black hole), and $\eta = Q/E^2$. Q is given by

$$Q = \rho_\theta^2 + \cos^2\theta \, [a^2(\xi^2 - E^2) + L_z^2\sin^{-2}\theta]. \tag{3}$$

It is readily verified that the radial motion of the particle depends on the following constants:

$$\hat{a} = a/M, \ \hat{E} = E/\mu, \ \hat{\xi} = \xi/M, \ \hat{\eta} = \eta/M^2,$$

where μ is the mass of the particle.

The radial motion of photons does not depend on the constant $\hat{E}$. Instead of the coordinate r, we now introduce $\hat{r}=r/M$. (The ^ - symbol will be omitted henceforth). Thus, the character of motion in the r-coordinate for the given value a is determined by the three constants E, ξ, η in the case of a moving particle, and by the two constants ξ and η in the case of photons.

Depending on the multiplicities of the roots of the polynomial $R(r)$ (for $r \geq r_+ = 1+\sqrt{1-a^2}$), we can have three types of motion in the r-coordinate (Zakharov, 1983; Zakharov, 1986), namely:

(1) the polynomial $R(r)$ has no roots (for $r \geq r_+$). The particle then falls into the black hole;

(2) the polynomial $R(r)$ has roots and $r_{max} > r_+$ (r_{max} is the maximum root); then we have $(\partial R/\partial r)(r_{max})>0$, and the particle departs to infinity after approaching the black hole;

(3) the polynomial $R(r)$ has a root and $R(r_{max}) = (\partial R/\partial r)(r_{max})=0$; the particle now takes an infinite proper time to approach the surface r=const.

We shall now examine the sets of motion E, ξ and η, corresponding to different types of particle motion for a given black hole rotation parameter a=const. Let us cut the space E, ξ, η with the plane E=const$\geq$1 and describe in this slice the set of constants corresponding to different types of motion. It then turns out that the boundary of the set of constants corresponding to the second type of motion for $\eta\geq0$ is the set of constants for which the motion belongs to the third type. We shall look upon this set as a graph of the function $\eta=\eta(\xi)$. We note that the set of these constants as functions $\xi(r)$ and $\eta(r)$ was examined by Chandrasekhar (1983). Some of the properties of the function $\eta(\xi)$ are described in the paper by Zakharov (1986).

There is a simple derivation of effective capture cross-section for particles possessing an arbitrary velocity at infinity in the field of the Schwarzschild black hole (Zakharov, 1985; 1988; 1991).

$$L^2_{cr} = \frac{-(d^2-18d-27)+(d+9)[(d+9)(d+1)]^{1/2}}{2d}, \quad \text{where } d=(E^2-1)^{-1}.$$

It should be noted that $\sigma = \pi L^2_{cr}/(E^2-1)$ is the particle capture cross-section (in units of square of the mass of the black hole).

Consider a moving particle of arbitrary energy at infinity ($E>1$). It can be verified that at

$$\eta_{max} = \frac{-(d^2-18d-27)+(d^4+28d^3+270d^2+972d+729)^{1/2}}{2E^2d}$$

$$r_{max} = (8d^3/27 + \eta_{max}E^2d(d/3+1))^{1/3}-2d/3 \qquad (4)$$

$$\xi_{max} = 2a/(r_{max}-2),$$

these values ensure that $R(r)$ and dR/dr vanish. We also note that for values chosen in accordance with (4) these values correspond to the maximum of $\eta(\xi)$. The values η_{max} and r_{max} turn out to be equal to the corresponding values of these quantities for a=0 (Schwarzschild metric) (Zakharov, 1985; 1991).

2. QUALITATIVE ANALYSIS OF SOME BOUND GEODESICS IN AN EXTREME KERR METRIC AND CONNECTION OF THE PROBLEM OF THE CLASSIFICATION OF PARTICLE MOTION IN THE KERR METRIC WITH THE SINGULARITIES OF THE SMOOTH FUNCTIONS (WITH "CATASTROPHE THEORY")

In this chapter it will be shown that a black hole in extreme rotation can have stable geodesics with any energy in the range $0\leq E \leq 3^{-1/2}\mu c^2$, where μ is the particle mass, although it is well known that for a particle moving along a circular geodesics in the field of a black hole in a state of extreme rotation, the binding energy has been found capable of reaching the value $3^{-1/3}\mu c^2$.

One readily finds (Carter, 1968) that if the particle is moving in the equatorial plane ($\theta=\pi/2$), Q=0. The particle will then travel in a circular orbit if the following relations

$$R(r)=0, \ \partial R/\partial r=0 \tag{5}$$

are satisfied, as well as the orbital stability criteria

$$\partial^2 R/\partial r^2<0. \tag{6}$$

Let us consider a black hole with an extreme value for the rotation parameter ($a=M$). Assume that the particle's orbits lie in the equatorial plane, with the following values for the constants of motion:

$$L_z=2E, \quad E<(1/\sqrt{3})\,\mu c^2, \quad r=M. \tag{7}$$

It is not hard to see that the criteria (5), (6) will then hold true. Accordingly, as a particle moves through successive near-circular orbits, an energy of order μc^2 can be liberated. It is also noteworthy that for any values of the constant Q ($Q>0$) and the constants of particle motion that satisfy the conditions (7), the criteria (5), (6) will remain valid; stable orbits will exist that correspond to the motion of a particle along the surface r=const with a given energy and given angular momentum, but which are not circular. The maximum value of $|\theta|$ for such orbits will be determined by the value Q. Outside the equatorial plane one finds, as Wilkins has argued, that stable nonequatorial orbits also can exist throughout the energy range $3^{-1/2}\mu c^2<E<\mu c^2$:

$$L_z=2E, \quad Q>3E^2-1. \tag{8}$$

One should recognize, however, that if the black hole departs slightly from a state of extreme rotation, there can be no stable circular orbits with energy in the range $0<E<3^{-1/2}\mu c^2$. In just the same way, the stable orbits (16) will vanish if the rotation falls short of the extreme case. This result can be demonstrated either by considering the potentials $V_\pm$, as done by Wilkins (Carter, 1968) (the potentials V_+, V_- will merge into a "knife edge" (Wilkins, 1972)), or by turning to Eqs (5), which implicitly specifies, for example, the parameters L_z and E as functions of the rotation parameter (Zakharov, 1986). The property that we have here is analogous to the straigth-line part of the relation between $\rho_\parallel$ ($\rho_\perp$) and the capture cross section in the case where $a=M$ and photons or particles (Zakharov, 1986) are incident on the black hole. Thus the presence of stable circular orbits is not the sole property that distinguishes extreme ($a=m$) from nonextreme holes. Since the Wilkins potentials coincide ($V_+=V_-$) for parameters whose values confirm criteria (6), one can readily see that the value m for the rotation parameter a represents a bifurcation point corresponding to "pleat" singularity (Brocker & Lander, 1975; Poston & Stewart, 1978; Arnol'd, 1984) and if a departs from that value, the stable orbits (7) will disappear.

We also showed the connection of the problem of classification particle motion in the Kerr metric with the singularities of the smooth functions (with "catastrophe theory"), particularly, those sets are connected with the semialgebraical submanifold of the algebraic manifold D_f ("swallow tail") (Zakharov, 1991), where the surface D_f (the "swallow tail") is defined in the space parameters u, v, w by the set (Brocker & Lander, 1975; Poston & Stewart, 1978; Arnol'd, 1984)

$$D_f=\left\{(u,v,w) \mid \exists x, \left\{ \begin{array}{l} x^4+ux^2+vx+w = 0 \\ 4x^3+2ux+v = 0 \end{array} \right\}\right\}.$$

Similarly, the algebraic curve on the manifold D_f is connected with a set of the photon motion constants, to which the multiple root of polynomial $R(r)$ corresponds (Zakharov, 1991).

3. CLASSIFICATION OF FINITE PARTICLE MOTION IN THE KERR METRIC

Here we shall consider only finite orbits for which the particle energy satisfies $E^2<1$. In this chapter we classify various types of finite motion of test particles in the Kerr metric by investigating the roots of $R(r)$, as Synge did for the Schwarzschild metric (Synge, 1960) and as we did for unbound particle motion in the Kerr metric.

We thus plan to classify particle motion having $E^2<1$. The polynomial $R(r)$ will then clearly have at least one root with $r>r_+=1+(1-a^2)^{1/2}$, since $R(r_+)>0$, and for large enough r, $R(r)<0$. Since for $r>r_+$ the polynomial can have no more than three distinct roots when multiplicity is taken into account (Zakharov, 1989), possible types of motion are as follows:

1) the polynomial $R(r)$ has one nonmultiple root at $r>r_+$. The particle will fall into the black hole in a finite proper time;

2) $R(r)$ has three nonmultiple roots $(r_+<r_1<r_2<r_3)$. There is then a range of r-values $(r_2<r<r_3)$ for which finite motion is possible over an infinitely long particle time. If $r_1<r<r_2$ or $r>r_3$, no motion is possible. If $r<r_1$, the particle will fall into the black hole in a finite proper time;

3) $R(r)$ has two distinct roots r_1 and r_2 with $r_+<r_1<r_2$ where r_1 is a nonmultiple root and r_2 is double. Particle motion is impossible for $r>r_2$ or $r_1<r<r_2$. There is a stable orbit at $r=r_2$, since $R''(r_2)<0$.

4) $R(r)$ has two distinct roots r_1 and r_2 with $r_+<r_1<r_2$ where r_1 is a double root and r_2 is a single root. Motion is impossible for $r>r_2$ and an unstable orbit exists for which $r=r_1$;

5) $R(r)$ has one triple root $r=r_1$ $(r_1>r_+$, $R(r_1)=R'(r_1)=0)$.

Types 3 and 4 are clearly manifolds of codimension 1, and type 5 is a manifold of codimension 2 in the space of constants defining the motion, namely E, L and Q. We shall refer to orbits for which particle motion is confined to a surface of constant r as being spherical. This terminology is not entirely accurate, but it is quite widespread.

Note that if the black hole is undergoing extreme rotation (a=1), the possible types of motion are as follows (with $L_z=2E$ and $E^2>1/2$):

1) type 4 for $0<Q<3(E^2-1/3)$;

2) for $Q=3(E^2-1/3)$ there is a triple root (r=1) and a root at $r_2>1$. Thus, there is an unstable orbit (type 6 motion) when r=1;

3) for $3(E^2-1/3)<Q<E^4/(1-E^2)$ we have a double root at r=1, as well as roots with $1<r_1<r_2$, so a region $r_1<r<r_2$ exists in which finite motion is possible and at r=1 we have stable spherical orbits (type 7 motion);

4) for $Q=E^4/(1-E^2)$ we have two double roots r=1 and $r = E^2/(1-E^2)$, and there are thus two stable spherical orbits (type 8 motion);

5) when $Q>E^4/(1-E^2)$, we have one double root (r=1) and thereby one stable spherical orbit (type 9 motion).

It should also be pointed out that just as for the corresponding orbits in the case of infinite particle motion, orbit types 6-9 disappear when the rotation parameter no longer has an extreme value (structural instability (Zakharov, 1989)). The case $1/3<E^2<1/2$ may be treated similarly.

There are thus two values of the angular momentum corresponding to stable circular orbits and two correspoding to unstable circular orbits for a given particle energy:

$$L^s=+\left\{\frac{-(a^2-18a-27)-(a+9)[(a+9)(a+1)]^{1/2}}{2a}\right\}, \tag{9}$$

$$L^u=+\left\{\frac{-(a^2-18a-27)+(a+9)[(a+9)(a+1)]^{1/2}}{2a}\right\}^{1/2}. \tag{10}$$

We shall give a description of the types of finite radial motion in the Schwarzschild metric. It is not difficult to verify that when the particle energy E is fixed and $|L|>|L^s|$, type 1 motion takes place; for $|L|=|L^s|$ we have type 3 motion, for $|L^u|<|L|<|L^s|$ we have type 2, for $|L|=|L^u|$ we have type 4 and for $|L|<|L^u|$ the motion is of type 1. Finally, $|L|=|L^u|=|L^s|$ results in type 5 motion (this accurately summarizes the well-known fact that the maximum of the polynomial $R(r)$ merges with the minimum at a=-9, E^2=8/9). We now investigate those sets of constants of the motion of a particle that correspond to different types of motion, given the value of the black hole rotation parameter (angular per unit mass). As before we pass a plane of constant E through E, L_z, Q space and examine the sets of constants that correspond to various types of motion in the Kerr metric. The results of studying those sets of constants of the motion of a particle that correspond to different types of motion are quoted in the papers of Zakharov (1989; 1991).

4. ON THE PHOTONS CAPTURE CROSS SECTION AND SLOW MOTION PARTICLE CAPTURE CROSS SECTION IN REISSNER-NORDSTROM METRIC

It is well known that the radial motion of a photon is determined by equation

$$\left(\frac{dr}{d\lambda}\right)^2 = R(r), \tag{11}$$

where $R(r) = r^4-\xi^2r^2+2\xi^2r-Q^2\xi^2$, Q - the charge of the black hole divided by the mass of the particle, $\xi=L/E$, L is the angular momentum of the photon and r is expressed in units of the mass of the black hole.

Thus the photon cross section of the black hole is defined by the expression (Zakharov, 1989; 1991a,b).

$$x^2=\frac{8q^2-36q+27+\sqrt{(8q^2-36q+27)^2+64q^2(1-q)}}{2(1-q)}, \tag{12}$$

where $q=Q^2$.

We also obtained a simple derivation for the capture cross-section given in the case of slow motion of the particle (Zakharov, 1990; 1991a,b). Analogous results are obtained in the relativistic theory of gravitation (Zakharov, 1990; 1991).

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